F_2 = $F_1/(SN^{0.9}D)$, feed rate term
$f(N)$ = effect of rotation speed on entrainment
$f(V)$ = effect of bed holdup on entrainment
h = thickness of bottom solid layer, m
I_e = intensity of eddy flow
K, K_1, K_2, k = proportionality constants
L = length of cylinder, m
N = cylinder rotation speed, rad/s
S = cylinder slope, m/m
U_t = average fines terminal velocity, m/s
U_x = gas velocity in the direction of cylinder axis, m/s
U_Y = eddy upward velocity, m/s
V = bed holdup volume per unit cylinder length, m^3/m
V_1 = $100\,V/(\pi D^2/4)$, holdup volume percentage
W = fines entrainment rate, kg/s
W_1 = $W/d(\pi D^2/4)$, m^3/s/m^2
X = fraction of cylinder volume occupied by solid bed
θ = dynamic angle of repose of solid, rad
ϕ = solid size distribution factor
ρ, μ = density and viscosity of gas, respectively

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Minimum Pressure in Dynamic Menisci

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The authors are interested in describing the properties of menisci, which are curved, free liquid-fluid interfaces in which surface tension effects are appreciable. This paper is concerned with the prediction of the location of the minimum pressure point in dynamic menisci, that is, in menisci distorted by flow. The geometry considered is free coating of flat sheets, as shown in Figure 1a. This geometry can be obtained in the laboratory using any one of several bath arrangements. One type of bath used to provide free coating is shown in diagram form in Lee and Tallmadge (1973b) and elsewhere.

The resultant profile of the liquid surface pressure P_s, as noted in Figure 1b, passes through a minimum at some point near the bath surface, often near the point where the interfacial curvature C is large. The normal stress boundary condition for this geometry is given by Batchelor (1967) as

$$\Delta P = P_0 - P_s = \sigma C + 2\mu \left(\frac{\partial u_s}{\partial s} \right) \tag{1}$$

Here

$$\sigma C \equiv \frac{\sigma}{R} \equiv \frac{\sigma\ \partial^2 h/\partial x^2}{[1 + (\partial h/\partial x)^2]^{3/2}} \tag{1a}$$

Here x is the vertical height. Based on Scriven (1960), it can be shown that Equation (1) applies where the following effects are negligible: surface viscosity and elasticity, surface tension gradients, and interphase mass transport; this is the case in studies with viscous oils by Lee and Tallmadge (1973a) and others.

The minimum pressure is of interest in order to characterize pressure profiles in dynamic menisci. The profiles themselves have been found useful in predicting the location of the free surface (Lee, 1973) and in calculating the relative contribution of the curvature and viscous terms in Equation (1) (Lee, 1973); furthermore, it is believed that a knowledge of profiles provide a better scientific description of menisci, stagnation points, and vortexes and may lead to a better understanding of nonuniformities in liquid

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coating processes. Measurement of minimum pressure is considered in the discussion section below.

For a static meniscus on a flat, vertical surface, the relevant differential equations involve force balance Equation (2) and normal stress Equation (3).

$$\frac{\partial P}{\partial x} + \rho g = 0 \qquad (2)$$

$$P_0 - P_s = \sigma C \qquad (3)$$

Equations (2) and (3) have been solved analytically to predict the location of the meniscus surface in terms of the profile of film thickness $h(x)$. One important property of the profile is the maximum height $a \equiv (2\sigma/\rho g)^{1/2}$, which is also the capillary rise for a wetting fluid.

Now consider the liquid pressure at the interfacial surface $P_s(x)$. The static meniscus results show that the surface pressure starts at the reference gas pressure P_0 at the bath surface and then decreases with height. From the static profile, it can be shown that the location of the minimum pressure point in a static meniscus occurs at the top of the meniscus, which is called the maximum height ($x_m = a$), at which the film thickness vanishes ($h = 0$).

The purpose of this note is to predict the location of the minimum pressure point as a function of velocity in free coating.

BASIC EQUATION

One long and complicated method of determining interfacial pressure P_s in free coating is to solve the Navier-Stokes equation in two-dimensional form. Such a study has been made (Lee, 1973). In this paper, however, we are concerned with obtaining a simple analytical expression, so an approximate method is used here. Approximating the flow in the thin film meniscus region as quasi-one-dimensional and neglecting inertial terms, the relationship for thickness h is given (Lee and Tallmadge, 1973b) by

$$3\mu u h^3\left(\frac{\partial P}{\partial x} + \rho g\right) = 3\mu u h_0 - \rho g h_0^3 \qquad (4)$$

Equation (4) was derived using the usual conditions of no slip at the wall and no drag at the interface and the well-known Landau-Levich procedure of integrating at constant height and equating two fluxes by continuity. One special case of Equation (4), which was found useful by Landau-Levich and others, can be formed by replacing the pressure term with an approximation for Equation 1a, namely that,

$$\partial P/\partial x = \sigma\, \partial^3 h/\partial x^3.$$

When Equation (4) is put into dimensionless form using $L \equiv h/h_0$, the resulting differential equation for $L(x)$, in terms of the pressure gradient, is

$$L^3 h_0 \frac{dP}{dx} = \mu u\,[3(L-1) - T_0^2(L^3 - 1)] \qquad (5)$$

Here $T_0 \equiv h_0(\rho g/\mu u)^{1/2}$, where T_0 is nondimensional parameter for film thickness; T_0 is not a function of x, but it is a function of the capillary number.

MINIMUM PRESSURE LOCATION

The location (L_m) of the minimum pressure can be obtained from Equation (5) by letting $dP/dx = 0$; thus we obtain

$$3(L_m - 1) - T_0^2\,(L_m^3 - 1) = 0 \qquad (6)$$

Equation (6) can be factored into the product of two terms as follows:

$$(L_m - 1)\,[3 - T_0^2(L_m^2 + L_m + 1)] = 0 \qquad (7)$$

However $L = 1$ is not the desired root because $h/h_0 = 1$ is simply the constant thickness region at the top (where $dP/dx = 0$). Thus the desired root is implied by the other factor in Equation (7), which is

$$T_0^2(L_m^2 + L_m + 1) - 3 = 0 \qquad (8)$$

Equation (8) has two roots, one positive and one negative. The positive root of Equation (8), obtained using the quadratic formula, is given by

$$L_m = \frac{-T_0 + (12 - 3T_0^2)^{1/2}}{2T_0} \qquad (9)$$

or

$$L_m = \left[\frac{3}{T_0^2} - \frac{3}{4}\right]^{1/2} - \frac{1}{2} \qquad (10)$$

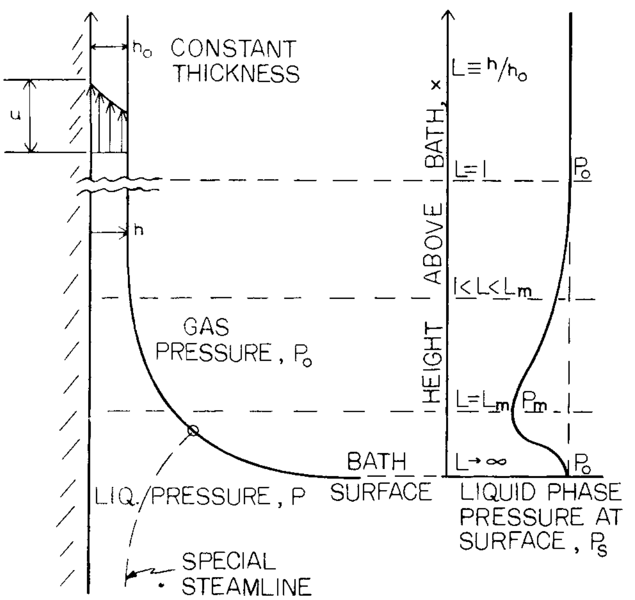

Fig. 1. Sketch of a pressure profile in free coating.

NUMERICAL EXAMPLES

In order to illustrate the prediction, a relationship between film thickness T_0 and capillary number, $Ca \equiv u(\mu/\sigma)$, is needed. One such expression (White and Tallmadge, 1965), which is valid within 1 to 20% for a wide range of Ca, is

$$T_0 = 0.944\, Ca^{1/6}\,[1 - T_0^2]^{2/3} \qquad (11)$$

The estimate of the pressure location $L_m \equiv h_m/h_0$ for various Ca is shown in Table 1. At low Ca, the minimum pressure point occurs at a large L_m; furthermore, L_m decreases at increasing speeds to a value between 1 and 2. Dimensional values of h_m are shown in Table 2, for a typical fluid with a viscosity of 0.10 N-s/m² (100 cp), a density of 1020 kg/m³ (1.02 gm/cc), and a surface tension of 0.040 N/m (40 dynes/cm).

It is not known from this analysis whether the minimum pressure point moves in terms of the laboratory height coordinate x. Table 2 does show that this minimum point occurs at a film thickness h_m which is very similar to the characteristic thickness h_c of the film. This provides some physical interpretation as to why the minimum pressure moves.

TABLE 1. THE LOCATION OF MINIMUM PRESSURE POINT

Speed, Ca	Film thickness, $T_0^{(a)}$	Location of the min. pressure, $L_m^{(b)}$	Location of the stagnation point, $L_s^{(c)}$
10^{-5}	0.1369	12.12	2.98
10^{-4}	0.1981	8.20	2.96
10^{-3}	0.2825	5.57	2.92
10^{-2}	0.3921	3.83	2.85
10^{-1}	0.5209	2.71	2.73
10^{0}	0.6524	2.01	2.57
10^{1}	0.7670	1.59	2.41
10^{2}	0.8534	1.34	2.27

(a) These values of film thickness $T_0(Ca)$ were obtained from the wide range, White-Tallmadge prediction of Equation (11). However, any tested function for $T_0(Ca)$ may be used.
(b) These values of pressure location, $L_m(T_0)$ were obtained using Equation (10) and the T_0 values given above.
(c) Calculated using Equation (12) and the T_0 values given above.

TABLE 2. FILM THICKNESS AT THE MINIMUM PRESSURE POINT
(0.100 N-s/m², 1020 kg/m³, 0.040 N/m)

Speed, Ca	Speed, u, mm/s	Char. h_c mm	Film h_0, mm	Min. pressure pt., h_m, mm	Min. press pt., x, mm
10^{-4}	0.04	0.2	0.04	0.3	Unknown
10^{-2}	4	2	0.8	3	Unknown
10^{0}	400	20	13	26	Unknown
10^{+2}	40,000	200	170	230	Unknown

Here $h_c \equiv (\mu u/\rho g)^{1/2}$, $h_0 \equiv T_0 h_c$ and $h_m \equiv L_m h_0$.

COMPARISON WITH OTHER SURFACE POINTS

The location of the stagnation point L_s is a second surface parameter of interest in dynamic menisci. It is useful for determining where surface downflow begins. It has been studied previously and is believed to vary with T_0 (and thus with Ca). For example, Lee and Tallmadge (1973b) gave the following prediction for the influence of speed on L_s:

$$L_s = 3 - T_0^2 \tag{12}$$

The estimate of the stagnation point location is also shown in Table 1. Comparison of L_s and L_m indicates that the location of the minimum pressure L_m occurs near the stagnation point for Ca near 10^{-1}, but is considerably larger in h_m (and lower in position x) at Ca below 10^{-2} than that for the stagnation point. Table 1 also suggests that the minimum pressure point may be much smaller in h_m (and higher in position x) at Ca above 10°.

The location of the junction point L_j is a third surface parameter of interest. It is based on linear regions of meniscus profiles, such as those observed experimentally and reported analytically by Lee and Tallmadge (1973a), using two expressions for the upper, thin region and the lower, thick region given by

$$\text{Thin} \quad \lambda \equiv x/h_0 = (M_1 \ln B_1) - M_1 \ln (L - 1) \tag{13}$$

$$\text{Thick} \quad \lambda \equiv x/h_0 = (M_2 \ln B_2) - M_2 \ln (L - 1) \tag{14}$$

where M_1, B_1, M_2 and B_2 are constants for each given meniscus profile.

The junction thickness is defined as the profile location where the two linear regions meet so that this thickness L_j may be used to state the experimentally observed delineations between the upper and lower regions of the film. For example, L_j was observed (Table 1 of Lee and Tallmadge, 1973a) to range from 2 to 3 in the Ca range of 0.4 to 24 for a viscous oil of 1.31 N-s/m².

Based on data near Ca of 10°, it was previously believed that the junction location was closely approximated by the stagnation point. More recent, but preliminary results, now suggest that the junction location may be more closely related to the minimum pressure location than to the stagnation point. However, more data are needed to test this hypothesis.

DISCUSSION

Direct measurement of the minimum pressure location L_m would probably be quite difficult. To avoid this, measurement of maximum curvature is suggested. For low Ca cases where the pressure jump is related to curvature by the Laplace equation $\Delta P = \sigma C$ [Equation (3)], the location of maximum curvature L_c is identical to the minimum pressure location. Thus

$$L_c = L_m \text{ at low } Ca \tag{15}$$

At higher Ca, a relationship between L_c and L_m can be written using the flow term in the normal-stress boundary condition, as shown in Equation (1).

In summary, the influence of speed and capillary number on the minimum pressure point is predicted here. In addition, the pressure point location is compared with other properties of meniscus profiles.

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NOTATION

a = capillary length, $(2\sigma/\rho g)^{1/2}$, mm
C = curvature, Equation (1a)
Ca = capillary number, $u(\mu/\sigma)$
h = film thickness in the meniscus, mm
h_0 = constant region thickness, mm
L = film thickness, nondimensional, h/h_0
L_c = location of the maximum curvature
L_m = location of the minimum pressure point
L_s = location of the stagnation point
P = pressure in the liquid phase, N/m²
P_0 = pressure in the gas phase, N/m²
s = distance along surface, mm
T_0 = film thickness, nondimensional, $h_0(\rho g/\mu u)^{1/2}$
u = coating speed, mm/s
u_s = surface velocity tangential to surface, mm/s
x = vertical coordinate, mm

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